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1989 J. Phys. A: Math. Gen. 22 L1167

(http://iopscience.iop.org/0305-4470/22/24/007)

J. Phys. A: Math. Gen. 22 (1989) L1167–L1171. Printed in the UK

LETTER TO THE EDITOR

Supersymmetric Bogomolny bounds at finite temperature

J Casahorran

Departamento de Fisica Teorica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

Received 29 August 1989

Abstract. We express the first quantum terms contributing to SUSY Bogomolny bounds at finite temperature. In particular the 'collective coordinates' treatment is mandatory in order to avoid an 'infrared catastrophe' associated with the bosonic partition function while the conventional saturation when $T=0$ is clearly recovered.

It is well known by now that in supersymmetric models the vacuum energy receives no quantum corrections. Passing to the soliton sector, Witten and Olive [1] showed that the conventional supersymmetry algebra includes central charges with a Bogomolny inequality directly derived from the proper algebra. Although this Bogomolny bound is saturated at the classical level it is interesting to analyse whether the same phenomenon happens at 'one-loop' order. In fact, the non-vanishing of the first quantum correction is traced to the existence of supersymmetric violating surface terms in the Lagrangian. In any case, the saturation of the bound at this level appears as a result of the absence of spontaneous breaking of $N=\frac{1}{2}$ SUSY in a related Lagrangian. The aim of this letter is to consider the finite-temperature effects over the Bogomolny bound. In particular we express the first quantum terms contributing to the inequality at finite temperature, using a 'collective coordinates' treatment in order to avoid an infrared catastrophe associated with the bosonic partition function. Moreover, the conventional saturation when $T=0$ is clearly recovered, a result very similar to the one obtained within the effective potential calculations: a term associated with the $T=0$ case and a second one which includes the finite-temperature contributions. We start from a general model governed by the action

$$S=\int\left(\tfrac{1}{2}(\partial_\mu\phi)^2+\tfrac{1}{2}\mathrm{i}\bar{\Psi}\gamma^\mu\partial_\mu\Psi-\tfrac{1}{2}W'(\phi)^2-\tfrac{1}{2}W''(\phi)\bar{\Psi}\Psi\right)\mathrm{d}^2x \qquad (1)$$

where ϕ is a real scalar field, while Ψ represents a Majorana spinor. The function $W(\phi)$ must be chosen such that the theory admits topological classical solutions, and the prime denotes a derivative with respect to the argument. In fact, the classical solutions $\phi_c(x)$ satisfy the Bogomolny equation

$$\frac{\mathrm{d}\phi_c(x)}{\mathrm{d}x}=\pm W'[\phi_c(x)] \qquad (2)$$

with a classical mass for the extended object obtained easily using (2)

$$M=\int_{-\infty}^{\infty}\left[\frac{1}{2}\left(\frac{\mathrm{d}\phi_c}{\mathrm{d}x}\right)^2+\tfrac{1}{2}W'(\phi_c)^2\right]\mathrm{d}x=|W(+\infty)-W(-\infty)|. \qquad (3)$$

If we shift the global field by the classical solution $\phi_c(x)$, i.e. $\phi(x)=\phi_c(x)+\varphi(x)$, these bosonic fluctuations $\varphi(x)$ obey the equation

$$\left(-\frac{d^2}{dx^2}+\tfrac{1}{2}(W'(\phi_c)^2)''\right)\varphi(x)=\omega_B^2\varphi(x). \tag{4}$$

Writing the spinor in its two-component form

$$\Psi(x)=\begin{bmatrix}u_+(x)\\ u_-(x)\end{bmatrix} \tag{5}$$

we obtain the following coupled equations

$$Q^+u_-=i\left(\frac{d}{dx}+W''(\phi_c)\right)u_-(x)=-\omega_F u_+(x) \tag{6a}$$

$$Qu_+=i\left(\frac{d}{dx}-W''(\phi_c)\right)u_+(x)=-\omega_F u_-(x) \tag{6b}$$

which, using the hidden SUSY quantum mechanics character of the Dirac equation over the background provided by $\phi_c(x)$, yield

$$Q^+Qu_+(x)=\left(-\frac{d^2}{dx^2}+W''(\phi_c)^2+\frac{dW''(\phi_c)}{dx}\right)u_+(x)=\omega_F^2u_+(x) \tag{7a}$$

$$QQ^+u_-(x)=\left(-\frac{d^2}{dx^2}+W''(\phi_c)^2-\frac{dW''(\phi_c)}{dx}\right)u_-(x)=\omega_F^2u_-(x) \tag{7b}$$

so that one of the fermionic components (it depends on the sign of (2)) satisfies the bosonic fluctuations, equation (4). With these data at hand, the first quantum correction to the mass is given by

$$\Delta M=\frac{1}{2}\sum(\omega_B-\omega_F). \tag{8}$$

Moreover, the conventional supersymmetry algebra which corresponds to (1) is modified by the emergence of central charges, namely [2]

$$\{Q_\alpha,\bar{Q}_\beta\}=-2\gamma^\mu_{\alpha\beta}P_\mu+i\gamma_3T \qquad \alpha,\beta=1,2 \tag{9}$$

with

$$\frac{T}{2}=\int_{-\infty}^{\infty}\frac{d}{dx}(W(\phi_c))\,dx. \tag{10}$$

In fact, T is different from zero only over the topological sectors of the theory and when evaluated at the classical level represents twice the mass (see (3)). If we take the rest frame, $P_\mu=(M,0)$ the SUSY algebra adopts the form

$$Q_1^2=2M-T \tag{11a}$$

$$Q_2^2=2M+T \tag{11b}$$

$$\{Q_1,Q_2\}=0 \tag{11c}$$

from which the quantum Bogomolny bound is easily derived

$$M\geqslant\tfrac{1}{2}|T|. \tag{12}$$

Now our attention is devoted to the analysis of (12) when the effects associated with the finite-temperature case are included. Starting from the energy contributions, the assumption is made that the conventional quantum fields φ and Ψ are to be described as systems in thermodynamical equilibrium at finite temperature $T=1/\beta$ (Boltzmann constant $k_{\mathrm{B}}=1$). Then the thermodynamics of the system is given by the partition function where its logarithm provides the Helmholtz thermodynamic potential [3]. Taking the ω_{B} (ω_{F}) bosonic (fermionic) eigenvalues over the background $\phi_{\mathrm{c}}(x)$, we pass to the respective partition functions. In fact

$$Z_{\mathrm{B}K}=\frac{\exp(-\beta\omega_{\mathrm{B}K}/2)}{1-\exp(-\beta\omega_{\mathrm{B}K})} \tag{13a}$$

for each bosonic degree of freedom and

$$Z_{\mathrm{F}K}=\exp(\beta\omega_{\mathrm{F}K}/2)[1+\exp(-\beta\omega_{\mathrm{F}K})] \tag{13b}$$

for each fermionic one. Then the bosonic (fermionic) free energies are given by

$$F_{\mathrm{B}}=\sum_k\left(\frac{\omega_{\mathrm{B}K}}{2}+\beta^{-1}\ln[1-\exp(-\beta\omega_{\mathrm{B}K})]\right) \tag{14a}$$

$$F_{\mathrm{F}}=\sum_k\left(-\frac{\omega_{\mathrm{F}K}}{2}-\beta^{-1}\ln[1+\exp(-\beta\omega_{\mathrm{F}K})]\right). \tag{14b}$$

Now we recall the stability equation associated with the bosonic part where an unavoidable zero-energy mode emerges due to translational invariance [4]. Although some results (such as the energy at 'one-loop' order) can be obtained without paying any special attention to this zero mode, higher orders appear full of divergences unless the zero-energy eigenmode receives a different treatment from the one outlined for the other modes. Passing to the finite-temperature case, we point out the infrared divergence which appears in (14a) when $\omega_{\mathrm{B}K}=0$. In order to avoid the problems we introduce a 'collective coordinates' treatment. In principle, we take the whole class of classical solutions parametrised by x_{c}, the centre-of-mass position of the soliton. Then we promote x_{c} in $\phi_{\mathrm{c}}(x-x_{\mathrm{c}})$ to a new quantum variable $X(t)$, precisely the 'collective coordinate' [4]. In this way we reach an unperturbed Hamiltonian which contains only non-zero frequencies. As we are discussing the energy eigenvalues in the $P=0$ frame the 'collective coordinate' method allows us to use a new sum Σ' where the prime denotes the zero-energy absence. Therefore we find for the global free energy

$$F=\sum_k\tfrac{1}{2}(\omega_{\mathrm{B}K}-\omega_{\mathrm{F}K})+\beta^{-1}\sum_k{}'\ln[1-\exp(-\beta\omega_{\mathrm{B}K})]-\beta^{-1}\sum_k\ln[1+\exp(-\beta\omega_{\mathrm{F}K})]. \tag{15}$$

The first term in (15) is the zero-point energy while the second term represents the finite-temperature contributions. In particular, the zero-temperature term admits an useful expression in accordance with simple relations between the respective spectral densities. Let us define n_+ (n_-) to be the densities of eigenfunctions of the operators Q^+Q (QQ^+). We suppose a Bogomolny bound which leads to

$$n_{\mathrm{B}}=n_+ \tag{16}$$

whereas the relation between n_{F} and $n_\pm$ reduces to [2]

$$n_{\mathrm{F}}=\tfrac{1}{2}(n_++n_-). \tag{17}$$

With these data at hand we can write a more rigorous version of the first quantum correction to the mass when $T=0$, namely

$$\Delta M_{T=0}=\frac{1}{4}\int\left(\frac{\mathrm{d}n_+(E)}{\mathrm{d}E}-\frac{\mathrm{d}n_-(E)}{\mathrm{d}E}\right)\sqrt{E}\,\mathrm{d}E \tag{18}$$

where the eigenvalues associated with the discrete spectrum cancel in a systematic way.

Resorting now to the general expressions which provide the differences of spectral densities, we have [5]

$$\Delta M_{T=0}=\frac{1}{4}\int -\frac{1}{2\pi E}\left(\frac{a_+\theta(E-a_+^2)}{(E-a_+^2)^{1/2}}-\frac{a_-\theta(E-a_-^2)}{(E-a_-^2)^{1/2}}\right)\sqrt{E}\,\mathrm{d}E \tag{19}$$

where $a_\pm=W''[\phi_c(x=\pm\infty)]$. Making the change of variable $E=k^2+a_\pm^2$, we obtain

$$\Delta M_{T=0}=-\frac{1}{4\pi}\int_0^\infty\left(\frac{a_+}{(k^2+a_+^2)^{1/2}}-\frac{a_-}{(k^2+a_-^2)^{1/2}}\right)\mathrm{d}k. \tag{20}$$

Later on we analyse the 'one-loop' term associated with the central charge. Using the shift $\phi(x)=\phi_c(x)+\varphi(x)$, the correction at this order adopts the form

$$\tfrac{1}{2}\Delta T=\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{2}W''(\phi_c)\langle\varphi^2(x)\rangle\right)\mathrm{d}x. \tag{21}$$

Working in the Euclidean frame, we simply recall the discretisation of the bosonic k_0 component at finite temperature

$$k_0=2\pi n/\beta. \tag{22}$$

Therefore (21) corresponds to

$$\tfrac{1}{2}\Delta T=-\frac{\beta}{8\pi^2}\sum_n\int\left(\frac{a_+}{n^2+v_+^2}-\frac{a_-}{n^2+v_-^2}\right)\frac{\mathrm{d}k}{2\pi} \tag{23}$$

with

$$v_\pm^2=\frac{\beta^2}{4\pi^2}(k^2+a_\pm^2). \tag{24}$$

Using the fact that

$$D(l,v)=(\pi/v)\coth\pi v\qquad\text{where}\qquad D(s,v)=\sum_n(n^2+v^2)^{-s} \tag{25}$$

we write

$$\tfrac{1}{2}\Delta T=-\frac{\beta}{8\pi^2}\int\left(\frac{a_+\pi}{v_+}\coth\pi v_+-\frac{a_-\pi}{v_-}\coth\pi v_-\right)\frac{\mathrm{d}k}{2\pi} \tag{26}$$

which can be reduced to

$$\begin{aligned}\tfrac{1}{2}\Delta T=&-\frac{1}{4\pi}\int_0^\infty\left(\frac{a_+}{(k^2+a_+^2)^{1/2}}-\frac{a_-}{(k^2+a_-^2)^{1/2}}\right)\mathrm{d}k\\&-\frac{1}{2\pi}\int_0^\infty\left(\frac{a_+\exp(-2\pi v_+)}{(k^2+a_+^2)^{1/2}(1-\exp(-2\pi v_+))}\right.\\&\left.-\frac{a_-\exp(-2\pi v_-)}{(k^2+a_-^2)^{1/2}(1-\exp(-2\pi v_-))}\right)\mathrm{d}k.\end{aligned} \tag{27}$$

So we have obtained the desired result: a first term which corresponds to the $T=0$ case and a second one including the finite-temperature effects. In particular, the Bogomolny saturation at zero temperature is clearly recovered (it suffices to consider (20) and (26)). Moreover, this saturation appears as a consequence of the $N=\frac{1}{2}$ supersymmetry which remains when expanding around the classical solution $\phi_c(x)$ [2]. We recall the $N=1$ case where the vacuum energy receives no quantum corrections. The last possibility considers the finite-temperature situation with automatic breaking of SUSY [6] and failure of Bogomolny saturation over the soliton.

Work supported by Comisión de Investigación Científico y Técnica (Spain).

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